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Supersymmetric 1+1D boundary field theory*

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Abstract
I discuss recent work with Anatoly Konechny proving a gradient formula for the boundary beta function of the general supersymmetric one-dimensional quantum system with boundary that is critical in the bulk but not at the boundary. I concentrate on some unanswered questions about which Aliosha expressed curiosity.

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One-dimensional quantum systems with boundary that are critical in the bulk but not at the boundary are characterized by their boundary coupling constants λ^a. The renormalization group leaves the bulk properties fixed, while the effective boundary coupling constants flow according to the boundary renormalization group equation

$$-T\frac{\partial\lambda^a}{\partial T} = \beta^a(\lambda) \tag{1}$$

(using the temperature T as the scale parameter).

Anatoly Konechny and I have proved a couple of general gradient formulae for the boundary beta functions $\beta^a(\lambda)$ [1, 2]. The gradient formula for the general boundary system is

$$\frac{\partial s}{\partial\lambda^a} = -g_{ab}(\lambda)\beta^b(\lambda), \tag{2}$$

where s is the so-called *boundary entropy,* and $g_{ab}(\lambda)$ is a certain positive-definite metric on the space of boundary systems, constructed from the two-point correlation functions (response functions) of the operators localized in the boundary. As a corollary,

$$-T\frac{\partial s}{\partial T} = \beta^a\frac{\partial s}{\partial\lambda^a} = -\beta^a g_{ab}(\lambda)\beta^b \leqslant 0 \tag{3}$$

establishing the second law of boundary thermodynamics—that the boundary entropy decreases with temperature, as it does in an isolated system whose entropy is $S = (1 + T\partial/\partial T)\ln\mathrm{tr}(\mathrm{e}^{-H/T})$, H being the Hamiltonian.

* In memory of Aliosha Zamolodchikov.

The second gradient formula pertains to the general *supersymmetric* one-dimensional system with boundary, critical in the bulk. The Hamiltonian of a supersymmetric system is of the form $H = \hat{Q}^2$, where $\hat{Q}$ is the supersymmetry generator. The supersymmetric boundary systems are characterized by the boundary coupling constants that preserve supersymmetry. We continue to write these as λ^a, in the context of supersymmetric systems. Changing scale preserves the supersymmetry, so the boundary beta functions are supersymmetric. The second gradient formula is

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}(\lambda)\beta^b(\lambda), \tag{4}$$

where z is the so-called *boundary partition function,* and $g^S_{ab}(\lambda)$ is a certain positive-definite metric on the space of supersymmetric boundary systems (not the same as the restriction of the first metric $g_{ab}(\lambda)$ from the space of all boundary systems). As a corollary,

$$-T\frac{\partial \ln z}{\partial T} = \beta^a \frac{\partial \ln z}{\partial \lambda^a} = -\beta^a g_{ab}(\lambda)\beta^b \leqslant 0, \tag{5}$$

so the *boundary energy* $T^2\partial \ln z/\partial T$ of the supersymmetric system is non-negative, as it is in an isolated supersymmetric system (whose thermodynamic energy is given by $T^2\partial \ln Z/\ln T = T^2\partial \ln \mathrm{tr}(\mathrm{e}^{-\hat{Q}^2/T})/\partial T = \langle \hat{Q}^2\rangle \geqslant 0$).

These gradient formulae should give some general control over the boundary rg flow, since they provide functions—s and $\ln z$—that decrease under the flow, and whose critical manifolds are the fixed points of the rg flow.

The gradient formulae are easily generalized to arbitrary quantum circuits that consist of bulk-critical quantum wires. A boundary is the simplest kind of junction in such a circuit. I have argued that such quantum circuits are ideal physical systems for asymptotically large-scale quantum computers [3]. The gradient formulae might be useful for general analysis of the computational power of such circuits.

The technical details of the proofs can be found in the papers [1, 2]. Here, I will only bring up some questions about the significance of the assumptions that the proofs rest on.

The boundary partition function z was constructed by Affleck and Ludwig [4, 5] by taking the full partition function $Z_L = \mathrm{tr}(\mathrm{e}^{-H_L/T})$ of a one-dimensional system of length L, in the limit of large L, then dividing by the universal partition function of the bulk system:

$$Z_L \sim \mathrm{e}^{LT\pi c/6}\, zz', \tag{6}$$

where c is the conformal central charge of the bulk-critical system. The remaining L-independent factors z and z' are the boundary partition functions of the two boundaries. They actually did this construction in the special circumstances where the boundary is also scale invariant, in which case the boundary partition function is a number independent of T, which they called g. The construction generalizes directly to boundaries that are not scale invariant. Affleck and Ludwig emphasized that $z = g$ is not the partition function of an isolated quantum system, pointing out that a genuine partition function in the limit $T \to 0$ goes to a positive integer, the ground state degeneracy, while, for critical boundaries, $z = g$ is typically not an integer, and can be less than 1.

The boundary entropy s is constructed from the entropy of the one-dimensional system by subtracting the universal bulk entropy per unit length:

$$S_L \sim \left(1 + T\frac{\partial}{\partial T}\right)\ln Z_L = L\frac{\pi c T}{6} + s + s'. \tag{7}$$

It is not obvious that this quantity s can be interpreted as the entropy of the boundary as a distinct sub-system. The second law of boundary thermodynamics, which follows from the gradient formula, gives some support to such an interpretation. In [3], I gave a general argument in support of this interpretation by writing a local, conserved *entropy current* operator which describes the flow of entropy in reversible processes within the quantum system. The change in s during such a reversible process is then given by the net entropy flow into the boundary. My hope is that the quantum entropy density and current operators will be useful tools for studying the information theoretic properties of large-scale quantum computers made from these near-critical quantum circuits.

The boundary second law and the entropy current formalism deal only with changes in the entropy s associated with the boundary. We have no handle on the total value of s. From Affleck and Ludwig's original observations, it is obviously not the case that $s \geqslant 0$. What is more disturbing, we cannot even say that s is bounded below. We cannot prove a universal lower bound on s, or a lower bound over the boundary conditions for a given bulk system, or even a lower bound on s as a function of temperature T for a given boundary system. Some partial results were found in [6]. It is implausible that an infinite amount of entropy could be extracted from the boundary of a one-dimensional quantum system at low temperature, but we have been unable to prove it. A lower bound on s is needed for control over asymptotic behavior of the rg flow. Without a lower bound on s, there is no way to be sure that the rg flow necessarily ends.

There is clearly a supersymmetric analog of the entropy flow formalism in which the super-partner of the entropy density is the super-charge density divided by T. Again, we can only control changes. We cannot put a global lower bound on $\ln z$ for supersymmetric boundary systems any more that we can on s for general boundary systems. We lack physical insight into the nature of s and $\ln z$.

The proofs of the gradient formulae depend on two technical assumptions: (1) bulk scale invariance, which implies bulk (super-)conformal invariance, and (2) a certain degree of ultraviolet regularity in response functions of operators at the boundary. I will display these assumptions in a direct proof of the positivity of the boundary thermodynamic energy.

The Hamiltonian and the super-charge are given by space integrals of the energy density and the super-charge density:

$$H = \int_0^L \mathrm{d}x\, \mathcal{H}(t,x) \qquad \hat{Q} = \int_0^L \mathrm{d}x\, \hat{\rho}(t,x). \tag{8}$$

These local fields are super-partners (components of the super-energy–momentum tensor):

$$[\hat{Q}, \hat{\rho}(t,x)]_+ = 2\mathcal{H}(t,x). \tag{9}$$

The local densities can be used to construct the boundary energy operator,

$$h(t) = \lim_{\epsilon\to 0} \int_0^\epsilon \mathrm{d}x\, \mathcal{H}(t,x), \tag{10}$$

and the boundary super-charge operator,

$$\hat{q}(t) = \lim_{\epsilon\to 0} \int_0^\epsilon \mathrm{d}x\, \hat{\rho}(t,x), \tag{11}$$

which are super-partners:

$$[\hat{Q}, \hat{q}(t)]_+ = 2h(t). \tag{12}$$

The boundary thermodynamic energy is given by

$$T^2 \frac{\partial \ln z}{\partial T} = \langle h \rangle. \tag{13}$$

We can separate the super-charge $\hat{Q}$ into boundary and bulk parts at $x = \epsilon$ just outside the boundary:

$$\hat{q}_\epsilon(t) = \int_0^\epsilon \mathrm{d}x\, \hat{\rho}(t,x) \qquad \hat{Q}_{\text{bulk}}(t) = \int_\epsilon^L \mathrm{d}x\, \hat{\rho}(t,x) \tag{14}$$

$$\hat{Q} = \hat{q}_\epsilon(t) + \hat{Q}_{\text{bulk}}(t). \tag{15}$$

Locality implies

$$[\hat{Q}_{\text{bulk}}(0), \hat{q}(0)]_+ = 0 \tag{16}$$

so

$$\langle 2h\rangle = \langle [Q, \hat{q}(0)]_+\rangle = \langle [\hat{q}_\epsilon(0), \hat{q}(0)]_+\rangle \tag{17}$$

but this equation is useless at $\epsilon = 0$ (where it formally implies positivity of the boundary energy), because $\langle [\hat{q}(t), q(0)]_+\rangle$ is ultraviolet divergent at $t = 0$.

We need a more subtle separation of the boundary from the bulk. We use the bulk super-conformal invariance to accomplish the separation. Define the Fourier transform

$$g_\epsilon(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle [\hat{q}_\epsilon(t), \hat{q}(0)]_+\rangle \tag{18}$$

and define response functions

$$G_\epsilon^\pm(\omega) = \pm \int_0^{\pm\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle [\hat{Q}_{\text{bulk}}(t), \hat{q}(0)]_+\rangle \tag{19}$$

analytic in the upper (resp. lower) half-plane. Now we have

$$2\pi\delta(\omega)\langle 2h\rangle = g_\epsilon(\omega) + G_\epsilon^+(\omega) + G_\epsilon^-(\omega). \tag{20}$$

It can be shown [2] that the bulk super-conformal invariance implies vanishing formulae:

$$G_\epsilon^+(\mathrm{i}\pi T) = 0 = G_\epsilon^-(-\mathrm{i}\pi T). \tag{21}$$

The vanishing formulae, along with the analyticity of the response functions, allow us to derive sum rules

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2} G_\epsilon^\pm(\omega) = 0 \tag{22}$$

as long as the $G_\epsilon^\pm(\omega)$ grow slow enough at large ω to allow us to deform the contour of integration to infinity. Applying the sum rule to equation (20) gives

$$\langle 2h\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2} g_\epsilon(\omega). \tag{23}$$

Now we can take $\epsilon \to 0$:

$$g(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle [\hat{q}(t), \hat{q}(0)]_+\rangle \tag{24}$$

as long as $\omega^{-2} g(\omega)$ is integrable at large ω. Finally, we obtain

$$\beta \frac{\partial \ln z}{\partial \beta} = -\beta\langle h\rangle = -\frac{\beta}{2} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2} g(\omega). \tag{25}$$

We have $g(\omega) \geqslant 0$ so

$$T \frac{\partial \ln z}{\partial T} \geqslant 0. \tag{26}$$

The ultraviolet conditions—on the large ω behavior of the correlation function $g(\omega)$ and of the response functions $G_\epsilon^\pm(\omega)$—are satisfied if the boundary physics is canonical in the ultraviolet. Then $\hat{q}(t)$ has ultraviolet scaling dimension $\leqslant 1/2$, so $g(\omega)$ and $G_\epsilon^\pm(\omega)$ have ultraviolet scaling dimensions $\leqslant 0$. The boundary behavior will certainly be canonical in the ultraviolet if the rg trajectory goes back to a fixed point at short distance. It is curious that the proof still works with ultraviolet behavior somewhat worse than canonical. We only need $\hat{q}(t)$ to have ultraviolet scaling dimension <1.

In the full proof of the gradient formula, ultraviolet regularity is similarly required in the two-point function of boundary operators. In particular, it is needed to construct the metric that appears in the gradient formula. In the supersymmetric case, the metric is given by

$$g^S_{ab} = \int \mathrm{d}\omega \frac{\pi T}{\omega^2 + \pi^2 T^2} f_{ab}(\omega), \tag{27}$$

where

$$f_{ab}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle [\hat{\phi}_b(t)\hat{\phi}_a(0)]_+ \rangle \tag{28}$$

is the super-response function of the fermionic boundary operators. We need $f_{ab}(\omega)/\omega^2$ to be integrable at large ω. Again, canonical ultraviolet behavior is more than sufficient. The $\hat{\phi}_a(t)$ then have scaling dimensions $\leqslant 1/2$, so $f_{ab}(\omega) = O(1)$ at large ω.

The boundary second law expresses a separation between the boundary sub-system and the bulk. The vanishing formulae (21) expressing bulk super-conformal invariance are infrared in character. The infrared vanishing formulae and the ultraviolet regularity of the response functions play purely technical roles in the proof. Why should the bulk (super-)conformal invariance be necessary for this separation of the boundary physics from the bulk? Perhaps it is not far-fetched that ultraviolet properties of the boundary should be relevant to the separation, but we do not understand in physical terms how this works. Again, there is a lack of physical insight into the separation of the boundary sub-system from the bulk.

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